

# Correlation matrices at the phase transition of the Ising model

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We set up an artificial dynamics for the 2-D Ising model by using the standard Monte Carlo dynamics. Using this dynamics we can establish a time series for each spin of the Ising lattice consisting of numbers, say 1 and  $-1$ . We can then consider the correlation matrix for all or part of these time series and study the distribution of its eigenvalues as a function of the number of time steps, the number and way the time series are chosen and the temperature of the Monte Carlo Process. Whereas at high temperatures we find, unsurprisingly, the Marchenko Pastur distribution known from Random matrix theory we find universal power law, except at very small and very large distances where we see finite size effects, near the critical temperature with an exponent that can be evaluated from known exponents.

*Introduction:* Correlation matrices have attracted attention for almost a hundred years starting with multivariate analysis in finance [1] and biology [2]. The latter introduced random matrices as a null hypothesis at this very early stage. Surprisingly this tool was almost not used in physics and in particular in the description of phase transitions. This is all the more surprising, as correlations are at the very heart of practically any analysis of phase transition. We shall show, that the correlation matrix and its eigenvalues can potentially be very useful in the analysis of critical phenomena. An analysis of the spectrum shows that near phase transition a power law appears in the high end of the eigenvalue distribution, while the standard Marchenko Pastur distribution appears when the temperatures are chosen far above the critical temperature. The power law can be inferred from the power law decay of spatial correlations, which is in strong contrast to the fast decay at high temperatures.

These properties are easily detected for long time series that can easily be obtained if we use Monte Carlo dynamics. In a practical context we often have to deal with short time series, particularly if stationarity limits the time horizon. By short we imply, that the number of time series used to construct the correlation matrix is larger than the length of the time series, which leads, due to the dyadic product structure of the correlation matrix, to singular correlation matrices. Therefore the number of nonzero eigenvalues is limited and cannot be increased by increasing the number of time series and therewith the dimension of the matrix. One may thus wonder if the additional time series have more information about the same system and what ways there exist to extract these. We propose here two ways that seem productive and sensitive to the question whether we are near the phase transition or not. The first is the use of a power

map as introduced in Refs. [9, 10] with the original purpose of reducing noise. In Ref. [11] it was shown, that this map is effective in breaking the degeneracy at zero eigenvalue and thus lifting the singularity of the matrix. With powers sufficiently near to 1 the spectrum so obtained is well separated from the original spectrum and has been termed *emerging spectrum*. In the same paper it has been shown for random matrix models, that the emerging spectrum is very sensitive to correlations. While the same holds true for the original spectrum, the emerging spectrum can hold many more eigenvalues, and thus present better statistics if the number of time series  $N$  is much larger than the length of the time series  $M$ . The second technique is more straightforward: We use the large number of time series to construct an ensemble of correlation matrices of dimension  $n < N$  but  $n > M$ . We now clearly have a much larger number of eigenvalues if we combine the spectra of the members of the ensemble thus not only improving statistics, but also gaining the opportunity to study variations around the mean. These options allow us to have sensitive tools that not only serve to study the phase transition on hand, but hold promise to be useful in other correlated cases, such as, e.g., non-equilibrium phase transitions, say in stationary systems or bifurcations, \*e.g., saddle node or pitchfork bifurcations, in non-linear dynamical systems.

We shall proceed first to derive the power law for the spectrum of the correlation matrix for the Ising model at the critical temperature and large arrays and to underscore the generality of this with result with numerical examples for fairly small but non/singular correlation matrices. Next, we shall analyze the case of singular correlation matrices presenting both the power map method and the ensemble method as described above. We close with an outlook of the potential of such methods.

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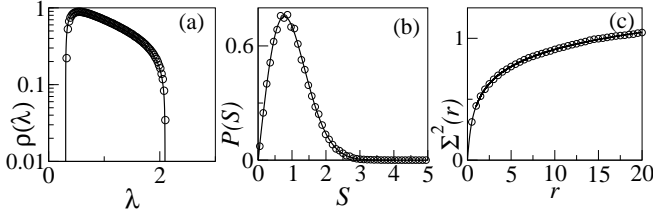


FIG. 1. Spectral statistics of  $C$  at a high temperature,  $2J/T = 0.001$ , for  $L = 192$ . In (a) we compare the spectral density with the Marčenko Pastur formula. In (b) and (c), we compare the nearest-neighbor-spacing distribution,  $P(S)$ , and the number variance,  $\Sigma^2(r)$ , with those of the RMT. Numerics are shown by open circles and theories are shown by solid lines.

*Scaling of The Correlation Spectrum at The Critical Point:* Here we shall provide a simple analytical calculation of the spectrum of correlation matrix at the critical temperature. Suppose we have a system on a regular cubic  $d$ -dimensional lattice of size  $L$ ,  $\mathbb{Z}_L^d$ , and assume that for asymptotically long times of observation  $T \rightarrow \infty$ , the correlations decay with exponent  $\alpha > 0$ ,

$$C_{\vec{n}, \vec{m}} \approx \frac{c}{|\vec{n} - \vec{m}|^\alpha}, \quad \text{for } 1 \ll |\vec{n} - \vec{m}| \ll L. \quad (1)$$

For the Ising model on a square lattice  $d = 2$ , we have for example  $\alpha = 1/4$ . If, in addition, periodic boundary conditions are assumed, then  $C_{\vec{n}, \vec{m}}$  is a  $d$ -dimensional *circulant* matrix, i.e.  $C_{\vec{n}, \vec{m}} = f(\vec{n} - \vec{m})$ , where  $f(\vec{n})$  is a function on a  $\mathbb{Z}_L^d$ . Therefore,  $C_{\vec{n}, \vec{m}}$  can be diagonalized in terms of a  $d$ -dimensional *discrete Fourier transformation*, yielding eigenvalues  $\lambda^{(\vec{k})}$  and eigenfunctions  $u_{\vec{n}}^{(\vec{k})}$ , labeled again by points  $\vec{k}$  on  $\mathbb{Z}_L^d$ .

$$\sum_{\vec{m}} f(\vec{n} - \vec{m}) u_{\vec{m}}^{(\vec{k})} = \lambda^{(\vec{k})} u_{\vec{n}}^{(\vec{k})}. \quad (2)$$

As the left-hand-side of eigenvalue equation (2) is a convolution, we obtain  $\lambda^{(\vec{k})}$  by transforming to momentum space  $u_{\vec{n}}^{(\vec{k})} = \sum_{\vec{l}} v_{\vec{l}}^{(\vec{k})} e^{-i\vec{l} \cdot \vec{n}}$ , namely

$$\lambda^{(\vec{k})} = \sum_{\vec{n}} f(\vec{n}) e^{i\vec{k} \cdot \vec{n}}, \quad (3)$$

and  $v_{\vec{l}}^{(\vec{k})} = \delta_{\vec{k}, \vec{l}}$ . For large  $L \gg 1$ , and given the asymptotic power-law scaling (1) of  $f(r) \approx cr^{-\alpha}$ , for  $1 \ll r \ll L$ , we can estimate the correlation eigenvalues for  $1 \ll |\vec{k}| \ll L$  as

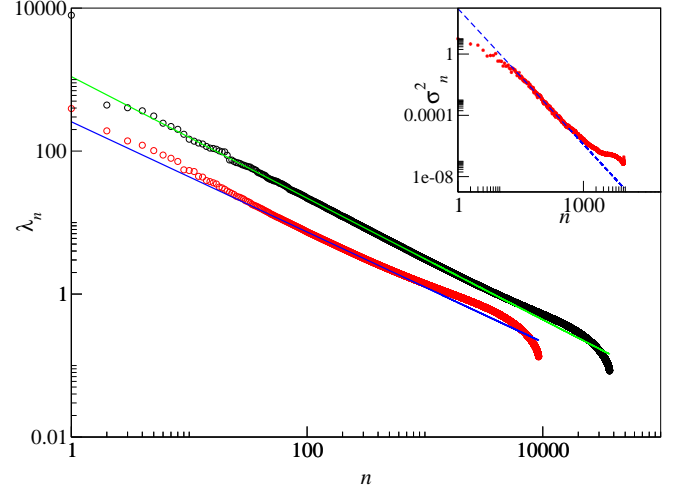


FIG. 2. (Color online) The Zipf plot for the eigenvalues,  $\lambda_n$ , of  $N \times N$  correlation matrices  $C$  at the critical temperature  $T_c$ . With *black* open circles we show eigenvalues for  $N = L^2$ . With *red* open circles we show ensemble averaged eigenvalues for  $N = L^2/4$ . In this figure  $L = 192$ . *Green* and *blue* lines are the power law fit where the exponent  $\gamma = 0.8504$  and  $0.771$ , respectively for the  $N = L^2$  and  $N = L^2/4$ . In both cases the time horizon  $M = 5N$ . In the inset we plot variances, with respect to the ensemble of 100 matrices of size  $L^2/4$ , of all the individual eigenvalues vs the eigenvalue-index. For the variances we again find a power law fit with an exponent  $\sim 2.95$ .

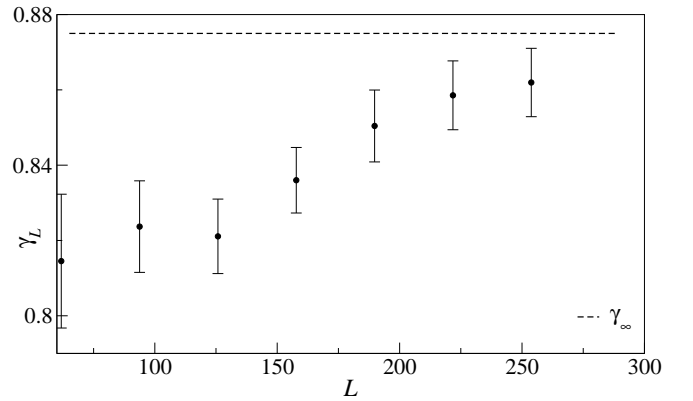


FIG. 3. The exponent  $\gamma$  calculated from the Zipf plot at the critical temperature vs the lattice dimension  $L$ .

$$\lambda^{(\vec{k})} \approx \int d^d \vec{r} \frac{c}{r^\alpha} e^{i\vec{k} \cdot \vec{r}} = \frac{c'_d}{|\vec{k}|^{d-\alpha}}, \quad (4)$$

where  $c'_d$  is a constant which in principle depends only on the dimensionality  $d$  and constant  $c$ . The last identity in (4) simply follows from non-dimensionalizing the integral. Labeling the eigenvalues by *decreasing eigenvalue*  $\lambda_j$ ,  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots$ , we find that  $j \propto |\vec{k}|^d$ , and

$$\lambda_j \approx \frac{c_d}{j^\gamma}, \quad \gamma = \frac{d-\alpha}{\alpha}, \quad \text{for } 1 \ll j \ll N = L^d. \quad (5)$$

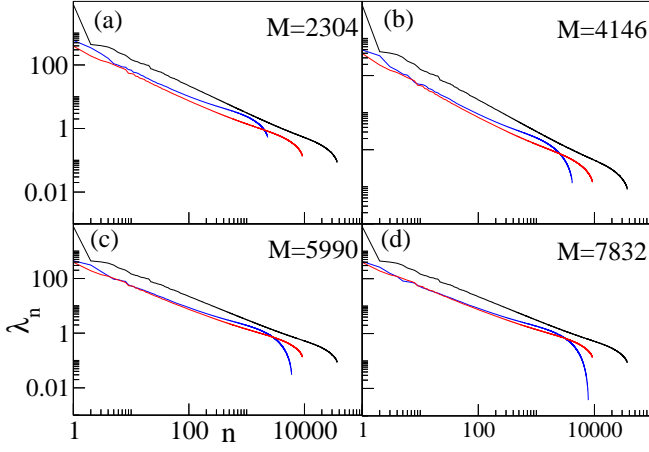


FIG. 4. (Color online) The Zipf plot at the critical temperature  $T_c$ . Here black lines represent eigenvalues for  $N = L^2$ , and red and blue lines represent averaged eigenvalues for  $N = L^2/4$ . For black and red lines we consider  $M = 5N$  while for the blue lines it is varied from  $M = 2304$  to  $M = 7832$ .

therefore the following scaling should hold for correlation eigenvalues at the critical temperature. For example, for 2d Ising model we have  $\gamma = \frac{7}{8}$ .

*Numerics for The 2d Ising Model:* In numerics, however, we consider two cases, viz., (i) we choose all the lattice sites and (ii) we choose only 1/4th of them at random. Note that the second case is not only cost efficient but one can have reasonably large number of matrices to improve the statistics. However, we first check the spectral statistics for the case (i), at a very high temperature, viz. when  $2J/T = 0.001$ . In our notations,  $J$  describes interaction with neighboring sites and  $T$  is the temperature. We indeed get RMT statistics at this temperature: As shown in Fig. 1 the density,  $\rho(\lambda)$  is closely described by the Marčenko-Pastur result [3, 11] and the fluctuations measures [4–6], viz. the nearest-neighbor-spacing distribution,  $P(S)$ , and the number variance,  $\Sigma^2(r)$ , as well coincide with those of the RMT [8]. In Fig. 2 we show the Zipf plot for the eigenvalues of  $N \times N$  dimensional  $C$ , for both cases, at  $T = T_c$  where  $2J/T_c = \ln(\sqrt{2} + 1)$ . The lattice size  $L = 192$ . In the case (ii) we consider an spectrum averaged over 100 realizations of  $C$ . In both the cases, the time horizon  $M = 5N$ . For the case (i), we obtain an averaged value,  $\gamma = 0.8504$ , for a range from  $\lambda_{100}$  to  $\lambda_{1000}$ . For the second case we still get the power law but with  $\gamma = 0.771$  for a range from  $\lambda_{115}$  to  $\lambda_{220}$ . Smaller eigenvalues in both cases instead have humps. This picture may be better understood from the peak in the spectral density near 0. For the uncorrelated case, such as we have at high temperatures, such peaks are well understood from the Dyson’s Brownian motion model [7] where the eigenvalues are treated as charged particles executing Brownian motion under the mutual logarithmic Coulombic repulsive potential, a positive confining potential and a negative logarithmic potential which prohibits the spectrum to have negative eigenvalues. The

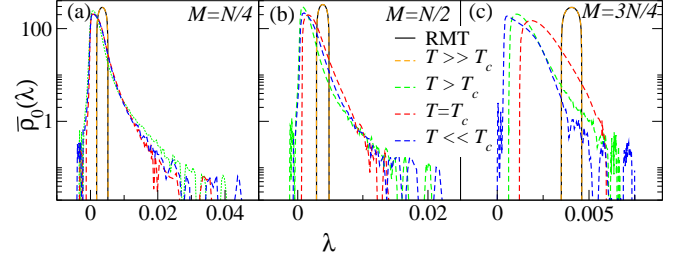


FIG. 5. (Color online) Ensemble averaged density of the emerging spectra,  $\bar{\rho}_0(\lambda)$ , under small-power map deformation where  $q = 1.001$ . In this figure densities are compared near the critical temperature and at a very high temperature with those of the RMM shown by solid black lines. Results shown from the left to right panels, respectively for  $M = N/4$ ,  $N/2$ , and  $3N/4$ .

latter is merely the consequence of the dyadic structure of  $C$ . The spectral density then represents an equilibrium. On the other hand, nonzero *nonrandom* off-diagonal correlations enhance such peak as could be seen in Fig. 2. In the inset of the same figure we also plot variances of each individual eigenvalue,  $\sigma_n^2$ , with respect to the eigenvalue-index  $n$ , for the case (ii). Interestingly, we again get a power law fit for the variances, in almost the same spectral range, with an exponent  $\sim 2.95$ . In Fig. 3, we show the exponent  $\gamma$  vs  $L$  to trace its convergence to  $\gamma_\infty$ . As is understood from the figure, the convergence of  $\gamma$  to  $\gamma_\infty$  is slow for large  $L$ . Next, we consider  $M < N$ . It is interesting to observe the power law even for short time horizon but for smaller range of spectra. As is evident from the Fig. 4, such ranges improves with  $M$ .

*The Emerging Spectra Near Criticality:* We now turn our attention to the power-map deformed matrices. The power map is defined for the matrix elements  $C_{mn}$ , with a highly nonlinear function [9–11],

$$C_{mn}^{(q)} = \text{sgn}[C_{mn}] |C_{mn}|^q \quad (6)$$

$$= C_{mn} \exp \left[ \frac{(q-1)}{2} \ln(C_{mn})^2 \right], \quad (7)$$

such that, for  $q = 1$  we get the same correlation matrix. The non-linearity of the power map breaks the dyadic structure of  $C$  and thus, for  $M < N$ , lifts degeneracy of eigenvalues at 0 resulting from the dyadic structure of  $C$ . For small  $q - 1$ , however, the spectrum emerged from the formally 0 eigenvalues is far-off the bulk spectrum. For the random matrix model (RMM), where  $C_{mn}$  is 1 at the diagonal and 0 on average for off-diagonal, the power map will displace those formally zero eigenvalues on average to  $\mathcal{O}(q - 1)$  and with the same amount the bulk spectrum but in the opposite direction [11]. For  $M \gtrsim N/8$ , almost all the eigenvalues of the emerging spectra are positive. In contrast for the correlated case, i.e near the critical temperature, even if  $M$  is slightly smaller than  $N$  negative eigenvalues will appear. This is seen in Fig. 5 where we show density of the emerging spectra,  $\bar{\rho}_0(\lambda)$ , for different but short time horizons

near the critical temperature. For this figure we consider  $L = 192$  and construct  $C$  using  $N = L^2/4$  randomly selected points. We first compare this density at a very high temperature, viz.  $2J/T = 0.001$ , for different  $M$ . This density nearly coincides with that for the RMM. Next, we compare densities near the critical temperature, ranging from  $2J/T = 2J/T_c - 0.01$  to  $2J/T = 2J/T_c + 0.01$ . Here, for small  $M$ , besides the differences in the densities, also notable are the significant portions of negative eigenvalues representing the Ising model near criticality.

*Conclusion:* Summarizing, we have two central points. First we show that in systems with power law spacial correlations we find a derived power law in the spectrum of the correlation matrix. While this is not entirely surprising we have not found previous use of the eigenvalues of the correlation matrices of critical periodic systems. Next, we use the 2d Ising model as the simplest example for numerical studies. Yet the analytic results are obtained for very long time series in the infinite limit.

In practice we often have to use short time series as stationarity of the processes involves may not be guaranteed over long time horizons. We therefore introduced standard Monte Carlo dynamics to be able to get numerical results for short time series and also far from criticality, where we can compare to random matrix models. The upshot is that the critical behaviour can also be seen with short time series or very incomplete correlation matrices, with randomly chosen sites for the individual time series. At high temperatures, unsurprisingly we find universal random matrix behaviour. Finally, we use techniques we recently developed to show high sensitivity to correlations in short time series to critical or near critical cases and we indeed see that the special correlation induce dramatic changes in what we call the emerging spectrum which results from lifting the degeneracy resulting from short time series with a power map with a power rather near to identity.

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